From last class,

$$V = \frac{mg}{K}\left(1 - C_4 e^{-k/m\, t}\right)$$

If $V(0) = V_0$, then

$$V_0 = \frac{mg}{K}\left(1 - C_4\right)$$

$$\Rightarrow \quad \frac{kV_0}{mg} = 1 - C_4$$

$$C_4 = 1 - \frac{kV_0}{mg}$$

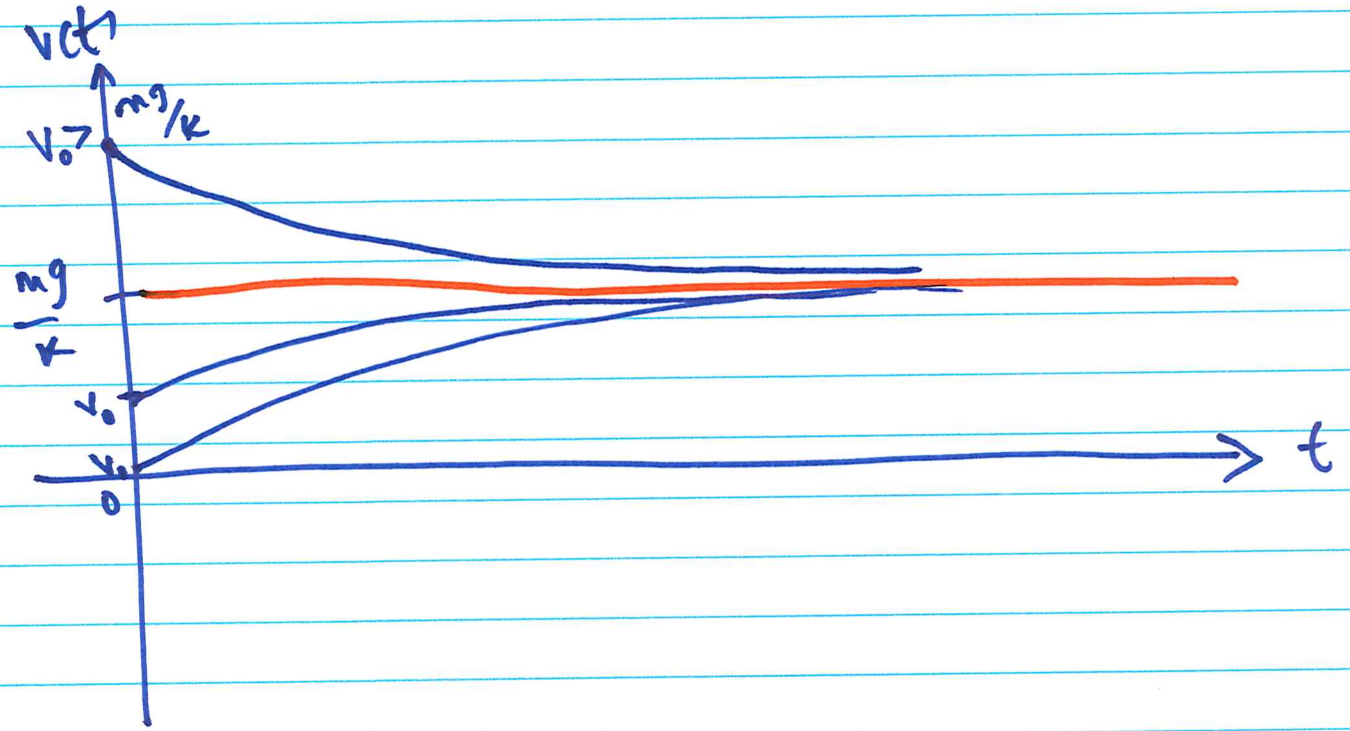
$$V = \frac{mg}{K}\left(1 - \left(1 - \frac{kV_0}{mg}\right) e^{-k/m\, t}\right)$$

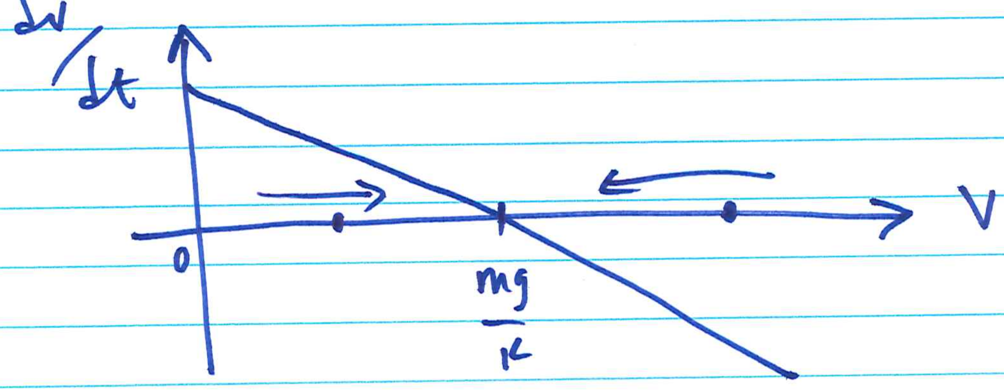
$$V = \frac{mg}{K}\left[1 - e^{-k/m\, t} + \frac{kV_0}{mg} e^{-k/m\, t}\right]$$

Take limit as $t \to \infty$

$$\Rightarrow \lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{mg}{k}\left[1 - e^{-k/m\, t} + \frac{kv_0}{mg} e^{-k/m\, t}\right]$$

$$\Rightarrow \quad v(t) = \frac{mg}{k} \quad \text{as } t \to \infty .$$





Review of partial derivatives

(i) Given $f(x,y) = x^2y - 4y^2 + 3xy$

$$\frac{\partial f}{\partial x} = 2xy + 3y \quad , \quad \frac{\partial f}{\partial y} = x^2 - 8y + 3x$$

(ii) $f(x,y) = 2y^3x + 3x^2y^2$

$$\frac{\partial f}{\partial x} = 2y^3 + 6xy^2 \quad , \quad \frac{\partial f}{\partial y} = 6y^2x + 6x^2y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2y^3 + 6xy^2\right) = 6y^2 + 12xy$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(6y^2x + 6x^2y\right) = 6y^2 + 12xy$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Surprised?!

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

These derivatives are called <u>mixed partials</u>.

Given a function $f(x,y)$ with differentiable partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

always holds, and this is referred to as <u>equality of mixed partials</u>.

$\ni$ - such that $\qquad \exists$ — there exist

Exact equations

Given an equation of the form

$$M(x,y) + N(x,y)\, dy/dx = 0$$

which can also be written as

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad (1)$$

Suppose $\exists$ a function $\psi(x,y)$ where $y \equiv y(x)$ $\ni$

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

Then we can write (1) as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d\psi(x, y(x))}{dx} = 0$$

(reversing implicit differentiation)

integrating,

$$\boxed{\psi(x, y) = \text{constant}}$$

This is the solution of the equation in (1) and equations of this form are called <u>exact equation</u>.

<u>Example:</u> Solve $2x+y^2 + 2xy\,y' = 0$

$$(2x+y^2) + (2xy)\frac{dy}{dx} = 0$$

$$\Rightarrow M(x,y) = 2x+y^2$$

$$N(x,y) = 2xy$$

$$\frac{\partial\psi}{\partial x} = 2x+y^2 \qquad \frac{\partial\psi}{\partial y} = 2xy$$

Integrating,

$$\psi(x,y) = x^2+xy^2+\gamma_1(y), \qquad \psi(x,y) = xy^2+\gamma_2(x)$$

(1) (2)

ff

Comparing 1 and 2

$\gamma_1(y) = 0$ and $\gamma_2(x) = x^2$

$\psi(x,y) = x^2 + xy^2$

The solution to the equation is

$$\psi(x,y) = c$$

$$\Rightarrow \quad x^2 + xy^2 = c$$

How do you determine if the ODE is exact?

Given an ODE,

$$M(x,y) + N(x,y)\, dy/dx = 0$$

By equality of partials, $\psi(x,y)$ exists if and only if

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

Provided ψ, $\frac{\partial \psi}{\partial x}$, $\partial \psi / \partial y$ are differentiable.

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

But $\frac{\partial \psi}{\partial y} = N$ and $\frac{\partial \psi}{\partial x} = M$

$$\Rightarrow \boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

This implies that for ψ to exist we need $N_x = M_y$ to hold.

Thus, the equation is exact if this condition holds.

Check

previous example:

$M(x,y) = 2x + y^2$, $N = 2xy$

$M_y = 2y \stackrel{?}{=} N_x = 2y$ ✓

Example: Solve $(3xy + y^2) + (x^2 + xy)\, dy/dx = 0$

$M = 3xy + y^2$, $N = x^2 + xy$

$M_y = 3x + 2y \neq 2x + y = N_x$

$\therefore$ not exact!!

Can we make the equation exact?

Yes! Multiply by an integrating factor.

INTEGRATING FACTOR

I.F. = integrating factor

Given an ODE

$$M(x,y) + N(x,y)\, dy/dx = 0$$

that is not exact, can ~~we~~ find an I.F. ~~such that~~ $h(x,y)$ $\ni$

$$h(x,y)M(x,y) + h(x,y)N(x,y)\, dy/dx = 0$$

is exact?

* I.F. approach is great but sometimes the function h can only be found for special cases.
* Simple I.F. can only be found where h is a function of only one ~~var~~ of the variables x and y, instead of both.
* An equation may have more than one I.F.

How to determine if an equation has an I.F of only one variable.

If $\dfrac{M_y - N_x}{N}$ is a function of x only,

then $\exists$ an I.F. that is a function of x only. And we solve the ODE

$$\frac{dh}{dx} = \left(\frac{M_y - N_x}{N}\right) h$$

to get h.

If $\dfrac{N_x - M_y}{M}$ is a function of y only

the $\exists$ an I.F that is a function of y only and we solve

$$\frac{dh}{dy} = \left(\frac{N_x - M_y}{M}\right) h$$ to get h.

See textbook for details...