Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\dfrac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$	(5)
$f'(t)$	$sF(s) - f(0)$	(6)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	(7)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(8)
$t^n \ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$	(9)
$\sin kt$	$\dfrac{k}{s^2+k^2}$	(10)
$\cos kt$	$\dfrac{s}{s^2+k^2}$	(11)
e^{at}	$\dfrac{1}{s-a}$	(12)
te^{at}	$\dfrac{1}{(s-a)^2}$	(13)
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	(14)
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	(15)
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	(16)
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	(17)
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$	(18)
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$	(19)
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	(20)
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$	(21)
$t\cosh kt$	$\dfrac{s^2-k^2}{(s^2-k^2)^2}$	(22)
$\delta(t-t_0)$	e^{-st_0}	(23)

Trig identities

$$\begin{aligned}\sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B\end{aligned}$$

This is probably the Table you will be given in your final exam.

Inverse Laplace transform

Given a function $Y(s)$ in frequency domain, we want to find a function $y(t)$ in time-domain such that

$$Y(s) = L[y(t)]$$

Example:

(1) $Y(s) = 7/s = 7 \cdot \frac{1}{s}$

$\Rightarrow y(t) = 7 \times 1 = \underline{\underline{7}}$

(2) $Y(s) = \frac{4s}{s^2+9} = 4 \frac{s}{s^2+3^2}$

$y(t) = 4\cos(3t)$

3) $$Y(s) = \frac{s}{(s+2)^2+4}$$

$$= \frac{s}{(s+2)^2+2^2} = \frac{s+2-2}{(s+2)^2+2^2}$$

$$= \boxed{\frac{s+2}{(s+2)^2+2^2}} - \boxed{\frac{2}{(s+2)^2+2^2}}$$

compare with (15) and (16) from our table.

$\Rightarrow \quad a=-2, \quad k=2$

$$\therefore y(t) = e^{-2t}\cos(2t) - e^{-2t}\sin(2t)$$

Example: (4) $$Y(s) = \frac{s^2+1}{s^2(s+2)}$$

$$\frac{s^2+1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

$\Rightarrow \quad A = -1/4, \quad B = 1/2, \quad C = 5/4$

$$Y(s) = -\frac{1}{4}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s^2} + \frac{5}{4}\cdot\frac{1}{(s+2)}$$

using our table,

$$y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{5}{4}e^{-2t}.$$

L.T. for ODE

Consider

$$y'' + y = 0 \qquad (1)$$

$$y(0) = 0, \quad y'(0) = 1 \qquad (2)$$

First, take L.T. of the ODE in (1)

$$L[y'' + y] = L[0]$$

$$L[y''] + L[y] = 0 \qquad (*)$$

$$L[y''] = \int_0^\infty y'' e^{-st}\, dt$$

We use integration ~~of~~ by parts twice

$$\int u\,dv = uv - \int v\,du$$

$$u = e^{-st} \Rightarrow du = -s e^{-st}\,dt$$

$$dv = y''\,dt \Rightarrow v = y'$$

$$L[y''] = y' e^{-st} \Big|_0^{\infty} - \int_0^{\infty} y' \left(-s e^{-st}\,dt\right)$$

$$= 0 - y'(0) + s \int_0^{\infty} y' e^{-st}\,dt$$

$$= -y'(0) + s\left[\left(y e^{-st}\right)\Big|_0^{\infty} + s\int_0^{\infty} y e^{-st}\,dt\right]$$

$$= -y'(0) - s\,y(0) + s^2 \underbrace{\int_0^{\infty} y e^{-st}\,dt}_{L[y]}$$

$$\therefore L[y''] = -y'(0) - s\,y(0) + s^2 L[y]$$

apply the initial condition

$$y(0) = 0, \quad y'(0) = 1$$

$$L[y''] = -1 + s^2 L[y]$$

put $L[y'']$ into $(*)$

$$-1 + s^2 L[y] + L[y] = 0$$

$$\Rightarrow \quad L[y] = \frac{1}{s^2+1}$$

using ~~the~~ our laplace table,

$$y(t) = \sin(t)$$

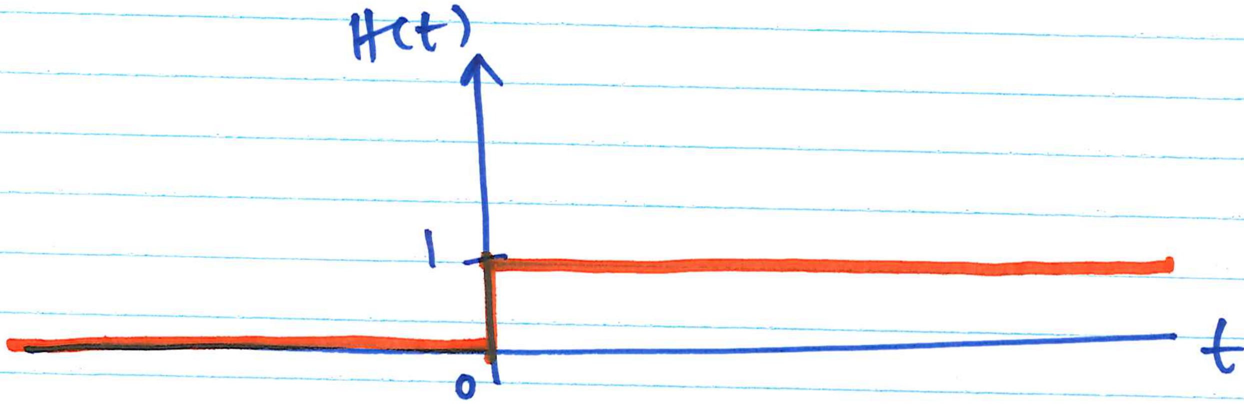
Consider the following problem

$$y'' + y = f(t) = \begin{cases} 4, & t > 1 \\ 0, & t \leq 1 \end{cases}$$

The forcing function for ~~for~~ this problem is called a step-function.

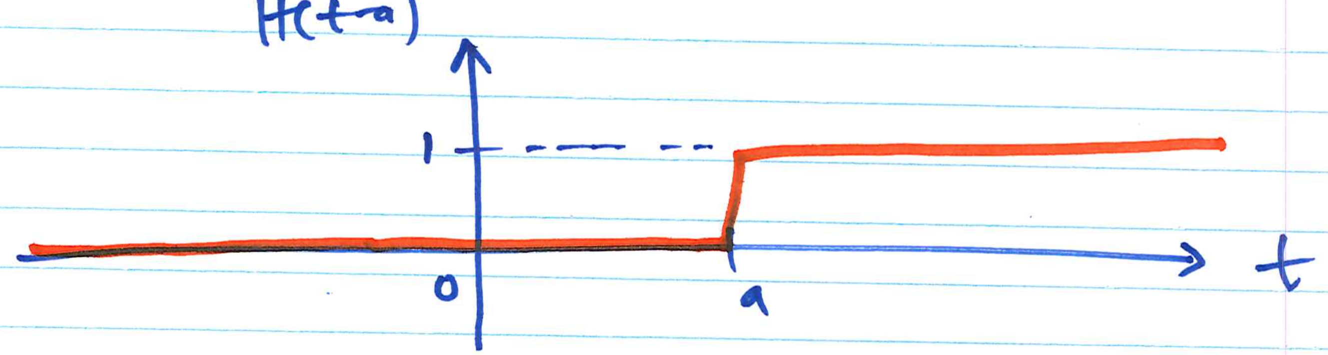
A typical example of a step function is the Heaviside function.

$$H(t) = \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0 \end{cases}$$



The step of the function can be shifted as desired,

$$H(t-a) = \begin{cases} 1, & t \geqslant a \\ 0, & t < a \end{cases}, \quad a > 0$$

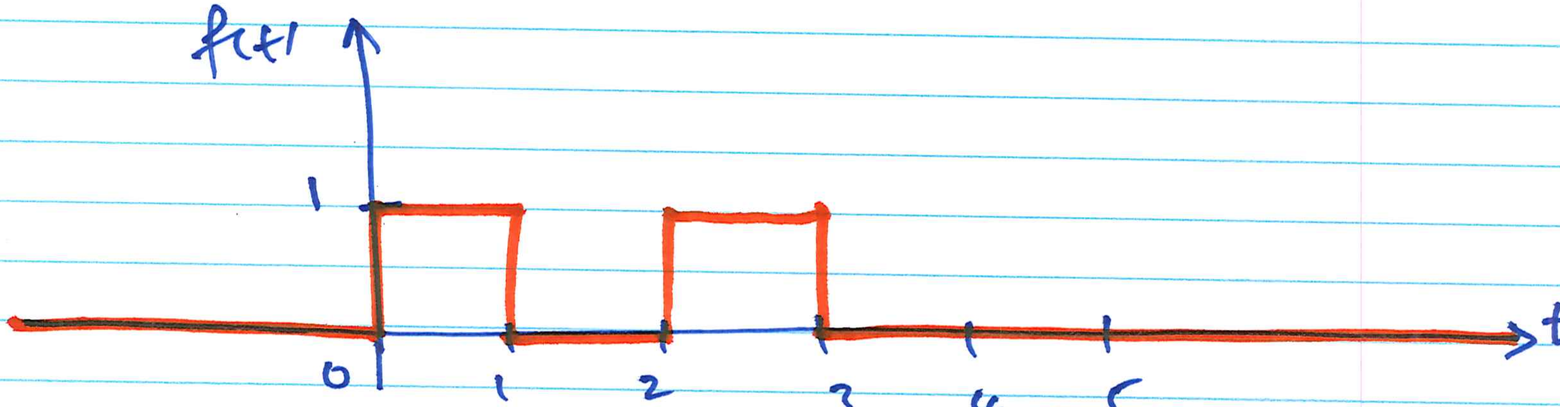


The function $u(t-a)$ on our table is called a "unit step function".

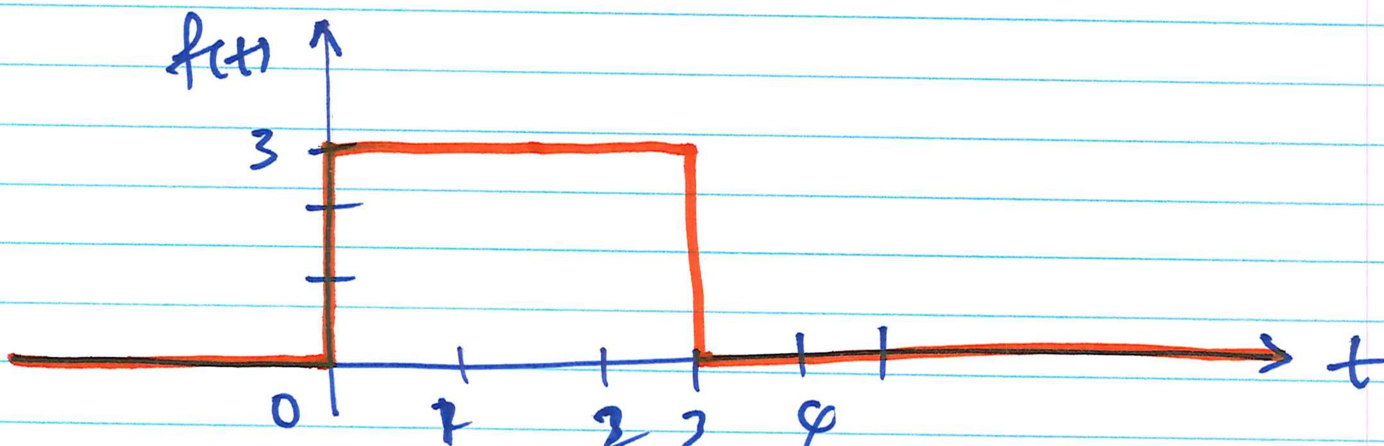
$$\Rightarrow \quad u(t-a) \equiv H(t-a)$$

Example: Draw the graph of the following functions:

(i) $f(t) = u(t) - u(t-1) + u(t-2) - u(t-3)$

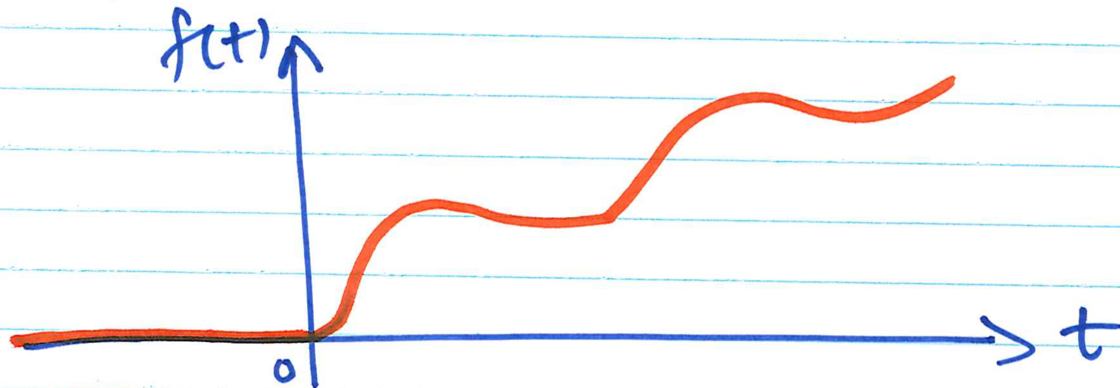


(ii) $f(t) = 3u(t) - 3u(t-3)$

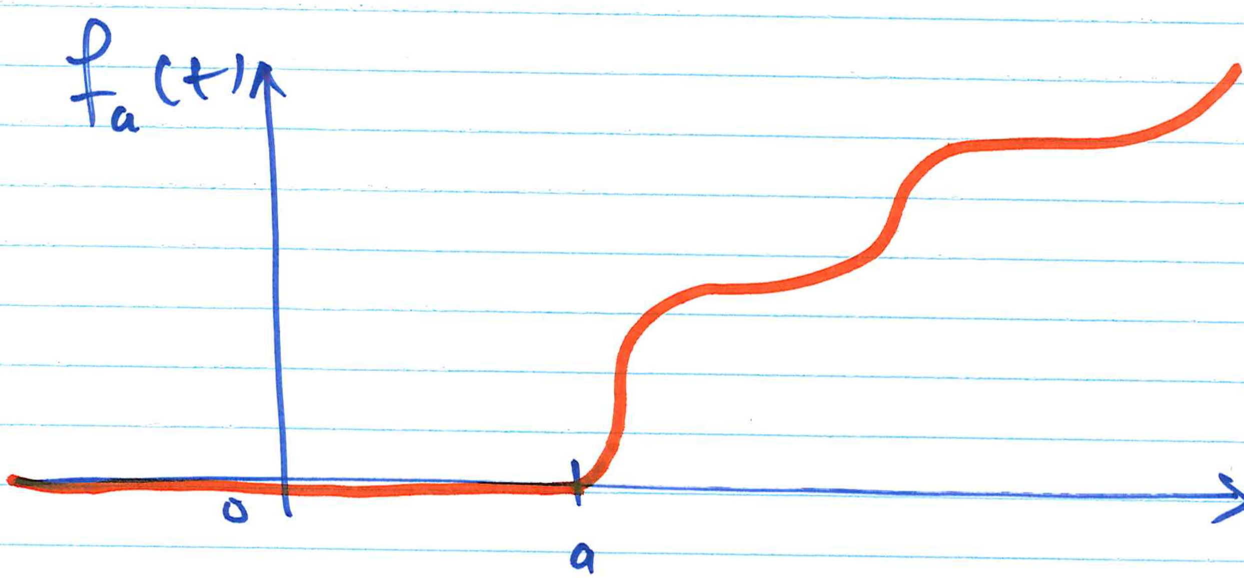


Shifted functions

Suppose a function $f(t)$ satisfies



Define $f_a(t) = u(t-a)\,f(t-a)$ as the a-shifted function of $f(t)$.



$$f_a(t) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases}$$