Last class

We started solving

$$\vec{y}\,'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$$

using the variation of parameter formula

$$\vec{y}(t) = \Psi \int \Psi^{-1} \vec{g}\, dt + \Psi \vec{c}$$

We got

$$\Psi = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix}$$

and

$$\Psi \int \Psi^{-1} \vec{g}\, dt = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t}$$

$\therefore$ The general solution of the system is

$$\vec{y}(t) = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t}$$

$$\vec{Y}(t) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t}$$

Observe that this solution is the same as what we got using the method of undetermined coefficients.

Second order constant coefficient ODEs

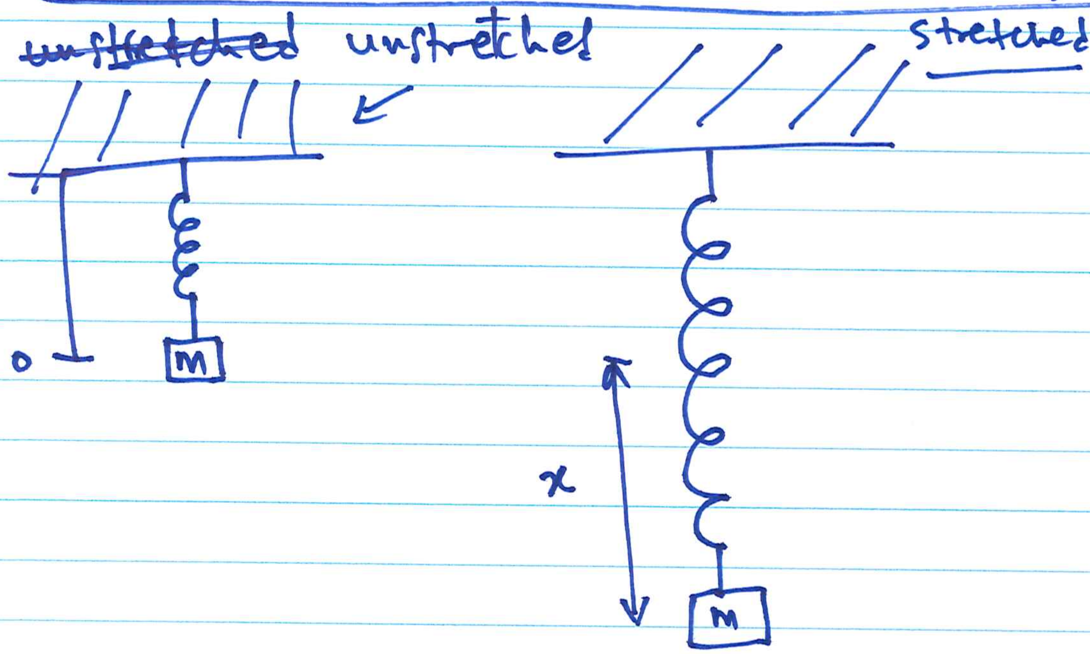
Consider

$$y'' + by' + cy = 0, \quad \text{Homogeneous (unforced)}$$

$$y'' + by' + cy = g(t), \quad \text{Non homogeneous (forced)}$$

where $g(t)$ is the forcing function.

1. Vibrating Springs (mass - spring)



From Hooke's law,

$$\text{restoring force} = -kx$$

where $k > 0$ is the spring constant.

From Newton's second law ~~[illegible]~~,

Sum of forces $= m\,a$

$$a = \frac{d^2 x}{dt^2}$$

$$-kx = m\frac{d^2 x}{dt^2}$$

- We have ignored external resisting forces.

$$\boxed{\frac{d^2 x}{dt^2} + \frac{k}{m}x = 0}$$ (undamped) simple harmonic motion

Let us introduce damping.

damping force $= -\mu V = -\mu \frac{dx}{dt}$

where $\mu > 0$ damping coefficient.

using Newton's 2nd law,

$$-kx - \mu \, dx/dt = m \frac{d^2x}{dt^2}$$

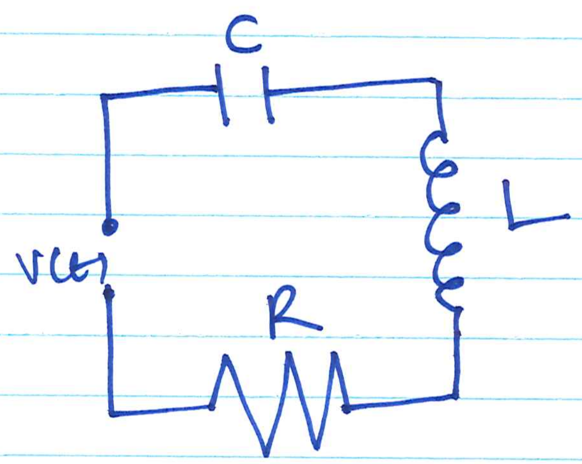
$$\frac{d^2x}{dt^2} + \mu/m \, dx/dt + k/m \, x = 0$$

let $\gamma = \mu/m$

$$\boxed{\frac{d^2x}{dt^2} + \underbrace{\gamma \, dx/dt}_{\text{damping term.}} + k/m \, x = 0}$$

② LCR Circuit

L – inductor, C – capacitor and R – resistor



Let I – current, φ – charge on capacitor

Voltage across the ;

(i) resistor, IR

(ii) inductor, $L\frac{dI}{dt}$

(iii) capacitor, φ/C

Kirchoff's 2nd law

sum of voltage drops = sum of applied voltage

$\Rightarrow$

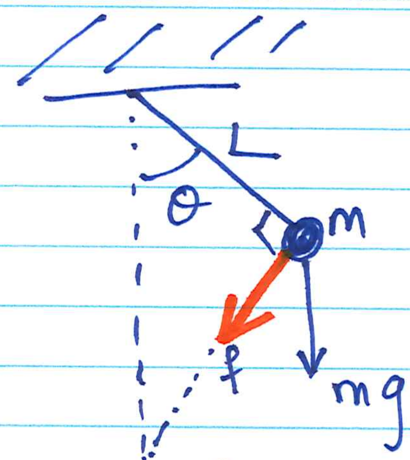
$$L\frac{dI}{dt} + IR + \frac{q}{C} = V(t)$$

But $I = \frac{dq}{dt}$

$$\Rightarrow \boxed{L\frac{d^2q}{dt^2} + \underbrace{R\frac{dq}{dt}}_{\text{damping term}} + \frac{q}{C} = \underbrace{V(t)}_{\text{forcing function}}}$$

damping term. forcing function

③ linear pendulum



consider the force in the direction perpendicular to the pendulum.

$$\sin(\theta) = \frac{f}{-mg}$$

$$f = -mg \sin(\theta).$$

~~acceleration~~, velocity,

$$V \propto \frac{d\theta(t)}{dt}.$$

$$V = L\frac{d\theta}{dt}$$

$$\Rightarrow \quad a = L\frac{d^2\theta}{dt^2}$$

Now, using Newton's second law ~~[illegible]~~

$$f = ma$$

$$-mg\sin\theta = mL\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

Suppose we are interested in small θ, then we can use the approximation

$$\sin\theta \sim \theta \quad \text{for small } \theta.$$

$\Rightarrow$ $$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$ ODE for a simple linear pendulum.

Now, consider the homogeneous equation

$$y'' + by' + cy = 0 \qquad \text{——— (1)}$$

~~Let~~ Let us write this equation as a system of first order ODE.

Let $\cancel{y_1 = y_1}$ $\cancel{y_2 = y_1'}$ $\qquad y_1 = y$

$\qquad y_2 = y'$

$$y_1' = y' = y_2$$

$$y_2' = y''$$

from (1), $y'' = -by' - cy$

$$y'' = -by_2 - cy_1$$

$y_1' = y_2$

$y_2' = -b y_2 - c y_1$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Following the same procedure, we can write higher order equations as a system of 1st order ODE.

Que: How do we solve the second order ODE?

$$y'' + by' + cy = 0 \quad \text{——— } (1)$$

Guess: $y(t) = e^{\lambda t}$, $y' = \lambda e^{\lambda t}$

put y into (1) $\qquad y'' = \lambda^2 e^{\lambda t}$

$$\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c e^{\lambda t} = 0$$

$$(\lambda^2 + b\lambda + c) e^{\lambda t} = 0.$$

$e^{\lambda t} \neq 0.$

$$\Rightarrow \quad \lambda^2 + b\lambda + c = 0$$

using the quadratic formula,

$$\lambda = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}$$

Forms of solution

① λ_1 and λ_2 are real and distinct.

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

② $\lambda_1 = \lambda_2 = \lambda$ (repeated roots)

$$\boxed{y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}}$$

③ λ is complex. Let $\lambda = \alpha \pm i\beta$.

$$y(t) = c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}$$

$$= e^{\alpha t}\left(c_1 e^{i\beta t} + c_2 e^{-i\beta t}\right)$$

$$= e^{\alpha t}\left(c_1 (\cos\beta t) + c_1 i\sin(\beta t) + c_2\cos(\beta t) - i\, c_2 \sin(\beta t)\right)$$

$$y(t) = e^{\alpha t}\left[(c_1+c_2)\cos(\beta t) + i(c_1-c_2)\sin(\beta t)\right]$$

Let $k_1 = c_1 + c_2$ and $k_2 = i(c_1 - c_2)$

$$y(t) = e^{\alpha t}\left[k_1 \cos(\beta t) + k_2 \sin(\beta t)\right]$$

$$\boxed{y(t) = k_1 e^{\alpha t}\cos(\beta t) + k_2 e^{\alpha t}\sin(\beta t)}$$