SYSTEM OF FIRST ORDER ODE

Suppose we have two dependent variables $x(t)$ and $y(t)$. We can write a system of equations for these variables,

$$\left.\begin{aligned} x'(t) &= f(t, x, y) \\ y'(t) &= g(t, x, y) \end{aligned}\right\} \quad \text{—— (1)}$$

Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\vec{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$

and $\vec{F}(t, \vec{x}(t)) = \begin{pmatrix} f(t, x, y) \\ g(t, x, y) \end{pmatrix}$

Then we can write (1) as

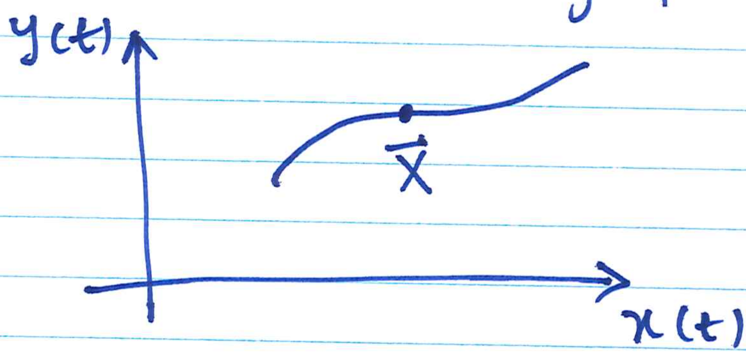
$$\vec{x}'(t) = \vec{F}(t, \vec{x}(t)) \quad \text{—— (2)}$$

The system is <u>autonomous</u> if

$$\vec{x}\,'(t) = \vec{F}\left(\vec{x}(t)\right)$$

That is, the system does not depend on t explicitly.

<u>Example</u>: Given the graph



$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{position vector.}$$

$$\vec{x}\,'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \quad \text{velocity vector}$$

So we can write an ODE system

$$\vec{x}\,'(t) = \vec{F}\left(t, \vec{x}(t)\right)$$

Suppose

$$f(t, x, y) = a(t)\, x(t) + b(t)\, y(t) + g_1(t)$$
$$g(t, x, y) = c(t)\, x(t) + d(t)\, y(t) + g_2(t)$$

Then the system in (1) becomes

$$\left.\begin{aligned} x'(t) &= a(t)\, x(t) + b(t)\, y(t) + g_1(t) \\ y'(t) &= c(t)\, x(t) + d(t)\, y(t) + g_2(t) \end{aligned}\right\} \text{— (2)}$$

which is a linear system of ODEs

Let

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \quad \left(\text{coefficient matrix}\right)$$

We can write (2) as

$$\vec{X}'(t) = A(t)\,\vec{X}(t) + \vec{G}(t) \quad \text{— (3)}$$

where $\vec{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $\vec{G}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$

If the matrix A is independent of t, i.e. A is a constant matrix. Then ~~the~~ we have a constant coefficient linear system.

The vector function $\vec{G}(t)$ is called the forcing function.

If $\vec{G}(t) = \vec{0}$, then (3) becomes

$$\vec{x}'(t) = A(t)\,\vec{x}(t)$$

and the system is called an homogeneous system, otherwise (3) is nonhomogeneous.

consider the matrix A. If $\exists$ a $\vec{v}$ such that

$$A\vec{v} = \vec{0} \text{ , then } \vec{v} \in Null(A)$$

(Null space of A)

$$\Rightarrow Null(A) = \{\vec{v} \mid A\vec{v} = \vec{0}\}$$

If $\vec{v} \in Null(A)$ and α is a scalar

then $\alpha\vec{v} \in Null(A)$

Let $\vec{x}_p$ be a particular solution to a linear algebraic system of equations, $A\vec{x} = \vec{b}$

Then $A\vec{x}_p = \vec{b}$

Let $\vec{v} \in Null(A)$ and α be a scalar

$$A(\vec{x}_p + \alpha\vec{v}) = A\vec{x}_p + \cancel{A(\alpha\vec{v})} \rightarrow \vec{0}$$

$$= A\vec{x}_p = \vec{b}$$

$$A(\vec{x}_p + \alpha\vec{v}) = \vec{b}$$

$\Rightarrow \quad \vec{x}_p + \alpha\vec{v}$ is the general solution of the algebraic system.

Now, consider the linear ODE

$$\frac{dy}{dt} + p(t)\,y(t) = g(t) \qquad (4)$$

Define the operator L (Linear operator)

$$L \equiv \frac{d}{dt} + p(t)$$

So that

$$L[y(t)] = \frac{dy(t)}{dt} + p(t)\,y(t)$$

$\therefore$ (4) can be written as

$$L[y] = g(t) \qquad (5)$$

Similar to system of algebraic equations, the general solution of (5) is given by

$$y_p(t) + c\,v(t)$$

where c is a constant.

$y_p(t)$ is one particular solution to

$$L[y] = g(t)$$

and

$v(t)$ is a solution to $L[v(t)] = 0$

Since $Null(L) = \{ v(t) \mid L[v(t)] = 0]$

we can say that $v(t) \in Null(L)$.

~~some~~ properties of operator L

(i) $L(f(t) + g(t)) = L(f(t)) + L(g(t))$

(ii) $L(c f(t)) = c L(f(t))$

(iii) $L(0) = 0$

In general, given a nonhomogeneous ODE

$$\frac{dy}{dt} + p(t)\, y(t) = g(t)$$

or

$$L[y(t)] = g(t)$$

Let $y_H(t)$ be the solution to the homogeneous problem

$$L[y_H(t)] = 0$$

and $y_p(t)$ be a solution to the nonhomogeneous problem

$$L[y_p(t)] = g(t).$$

$\therefore$ The general solution of the ODE is

$$y(t) = c\, y_H(t) + y_p(t).$$

where C is a constant

Example: consider

$$\frac{dy}{dt} + \frac{2y}{t} = \frac{\sin(t)}{t^2} \quad \left(\text{from quiz } 1\right)$$

with general solution

$$y(t) = \frac{c}{t^2} - \frac{\cos(t)}{t^2} \quad \text{———} \quad (*)$$

consider the homogeneous problem

$$\frac{dy}{dt} + \frac{2y}{t} = 0$$

$$\frac{dy}{dt} = -\frac{2y}{t}$$

$$\int \frac{dy}{y} = \int -\frac{2}{t}\,dt + C_1$$

$$\ln y = -2\ln t + C_1$$

$$y(t) = e^{\ln\left(\frac{1}{t^2}\right) + C_1} = C_2\,\frac{1}{t^2}$$

$$\Rightarrow \quad y_H(t) = \frac{C_2}{t^2}$$

$$\therefore \text{ from } (*), \quad y_p(t) = -\frac{\cos(t)}{t^2}$$

$$\Rightarrow \quad y(t) = y_H(t) + y_p(t) = \frac{C_2}{t^2} - \frac{\cos(t)}{t^2}$$