Example: Find the inverse L.T of

$$Y(s) = \frac{6}{s(s^2+9)}$$

let

$$\frac{6}{s(s^2+9)} = \frac{A}{s} + \frac{Bs + C}{s^2+9}$$

$$= \frac{As^2 + 9A + Bs^2 + Cs}{s(s^2+9)}$$

collect coefficients,

For s^2: $A + B = 0 \Rightarrow A = -B$

For s : $C = 0$

For constants : $9A = 6 \Rightarrow A = \frac{2}{3}$

$\Rightarrow B = -2/3$

$$\therefore \frac{6}{s(s^2+9)} = \frac{2}{3} \cdot \frac{1}{s} + \left(-\frac{2}{3}\right) \frac{s}{s^2+9}$$

$$L^{-1}[Y(s)] = \frac{2}{3} - \frac{2}{3} \cos(3t)$$

CONVOLUTION

Given two functions $f(t)$ and $g(t)$, the convolution of the two function is given by

$$\boxed{(f*g)(t) = \int_{-\infty}^{\infty} f(t-x)\, g(x)\, dx}$$

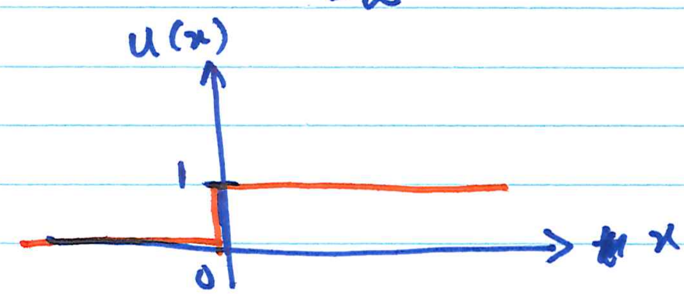
If $t \geq 0$, then we can write

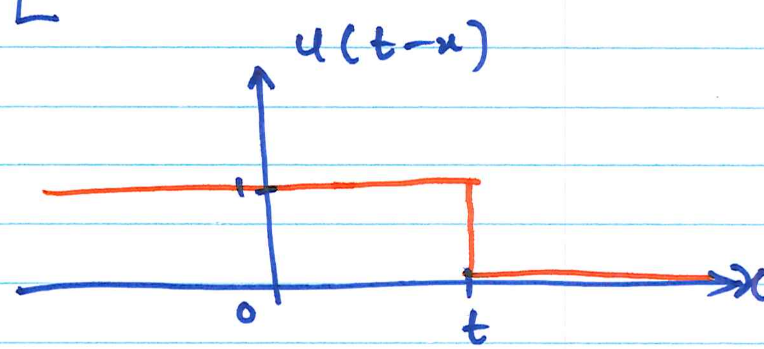
$f(t)$ as $f(t)\, u(t)$

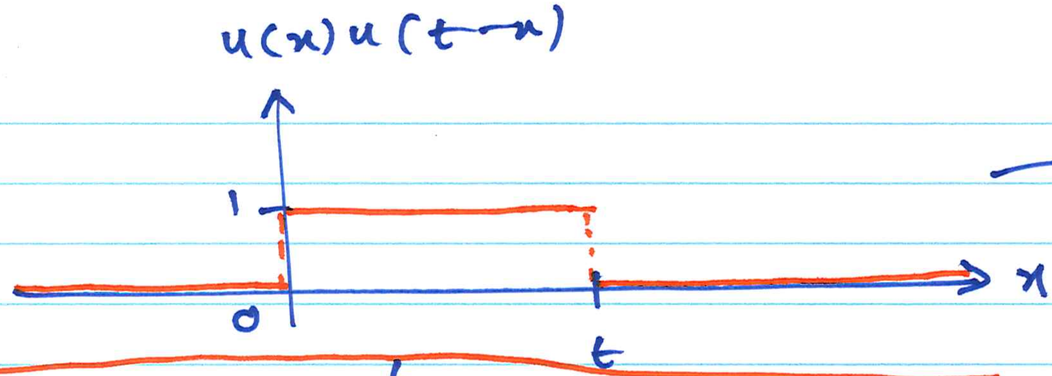
$g(t)$ as $g(t)\, u(t)$

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-x)\, u(t-x)\, g(x)\, u(x)\, dx$$

$$= \int_{-\infty}^{\infty} f(t-x)\, g(x) \left[u(t-x)\, u(x) \right] dx$$







(graph 3)

$$\boxed{(f*g)(t) = \int_0^t f(t-x)\, g(x)\, dx \ , t \geq 0}$$

We can view convolution as an operation on two functions $f(t)$ and $g(t)$ to produce another function

$$h(t) = (f*g)(t) .$$

<u>Example:</u> Find $(f*g)(t)$ for $f(t) = u(t)$ and $g(t) = 2u(t)$.

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-x)\, g(x)\, dx$$

$$= \int_{-\infty}^{\infty} u(t-x) \cdot 2u(x)\, dx = 2\int_{-\infty}^{\infty} \left[u(t-x)\, u(x)\right] dx$$

using graph 3 above,

$$(f*g)(t) = 2\int_0^t 1\,dx = 2\left[x\right]_0^t = 2t.$$

$$\therefore (f*g)(t) = 2t$$

<u>**Example:**</u> Find the convolution of $f(t) = t$ and $g(t) = \sin(t)$, $t \geq 0$.

$$(f*g)(t) = \int_0^t f(t-x)\,g(x)\,dx$$

$$= \int_0^t (t-x)\sin(x)\,dx$$

$$= \int_0^t t\sin(x)\,dx - \underbrace{\int_0^t x\sin(x)\,dx}_{\text{integration by parts.}}$$

$$(f*g)(t) = t\left[-\cos(x)\right]_0^t - \left[-t\cos(t) + \sin(t)\right]$$

$$(f*g)(t) = t - \sin(t)$$

Some properties of convolution

(i) $(f*g)(t) = (g*f)(t)$ (commutativity)

(ii) $(f*(g_1+g_2)) = (f*g_1) + (f*g_2)$

(distributive law)

(iii) $(f*g)*h = f*(g*h)$ (associativity)

(iv) $f*0 = 0*f = 0$

L.T of convolution of two functions

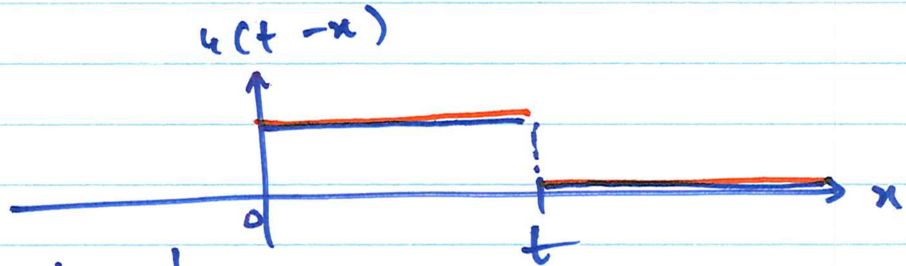
Suppose $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$

$$L[(f*g)(t)] = \int_0^\infty (f*g)\, e^{-st}\, dt$$

$$= \int_0^\infty \left(\int_{-\infty}^\infty f(t-x)\, g(x)\, dx \right) e^{-st}\, dt$$

$$= \int_0^\infty \left(\int_0^t f(t-x)\, g(x)\, dx \right) e^{-st}\, dt$$

$u(t-x)$



consider the inner integral,

$$\int_0^t f(t-x)\, g(x)\, dx = \int_0^\infty f(t-x)\, u(t-x)\, g(x)\, dx$$

$$\therefore\ L\left[(f*g)(t)\right] = \int_0^\infty \left(\int_0^\infty f(t-x)\, u(t-x)\, g(x)\, dx \right) e^{-st}\, dt$$

$$= \int_0^\infty g(x) \left(\int_0^\infty \left[f(t-x)\, u(t-x) \right] e^{-st}\, dt \right) dx$$

$$= \int_0^\infty g(x)\, L\left[f(t-x)\, u(t-x) \right] dx$$

$$= \int_0^\infty g(x)\, e^{-xs}\, F(s)\, dx$$

$$= F(s) \underbrace{\int_0^\infty g(x)\, e^{-xs}\, dx}_{G(s)}$$

$$\boxed{L[(f*g)(t)] = F(s)\cdot G(s)} \text{ — (box 1)}$$

This implies that the L.T. of convolution of two functions $f(t)$ and $g(t)$ is equal to the product of the L.T. of $f(t)$ and that of $g(t)$.

Return to our first example!

$$Y(s) = \frac{6}{s(s^2+9)} = \frac{2}{s}\cdot\frac{3}{s^2+9}$$

$$= \underbrace{L[2]}_{f}\cdot \underbrace{L[\sin(3t)]}_{g}$$

From box 1, we have

$$L[(f*g)] = L[2]\cdot L[\sin(3t)]$$

$$L^{-1}[Y(s)] = (f*g)(t) = \int_0^t 2\sin(3\tau)\,d\tau$$

$$y(t) = 2\left[-\frac{\cos(3\tau)}{3}\right]_0^t = -\frac{2}{3}\cos(3t) + \frac{2}{3}$$

NB: The same as what we obtained using partial fractions.