Autonomous nonlinear systems

consider

$$\vec{Y}' = \vec{F}(\vec{Y})$$

where $\vec{F}$ is nonlinear.

Usually these types of systems are not solvable analytically - But ~~can~~ we can;

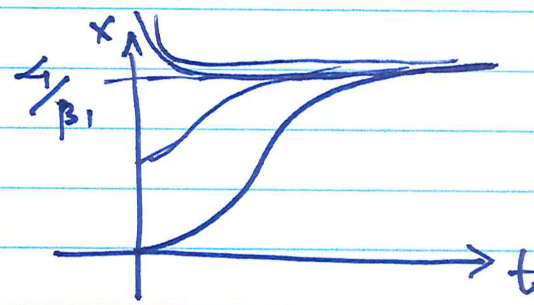
* solve them numerically
* plot their vector fields and use it to study the qualitative ~~behav.~~ behaviour of the system.
* Study the qualitative behaviour of the system using linear analysis.

Example: competing species model.

Let $X(t)$ and $Y(t)$ be population of two competing species.

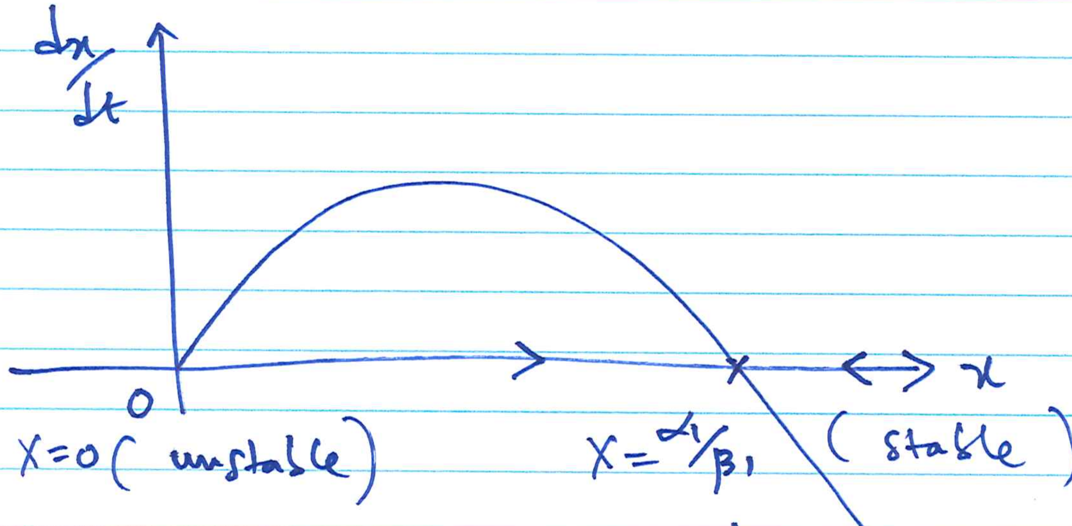
Suppose in the absence of species y, $x(t)$ satisfies

$$\frac{dx}{dt} = x(\alpha_1 - \beta_1 x)$$



$\Rightarrow$ $X = 0$ and $X = \alpha_1/\beta_1$

are steady-state solutions of this model



Similarly, in the absence of species x, $y(t)$ satisfies $\frac{dy}{dt} = y(\alpha_2 - \beta_2 y)$

$\Rightarrow$ $y = 0$ (unstable), $y = \alpha_2/\beta_2$ (stable).

Let us couple the two ~~models~~ equations;

$$\frac{dx}{dt} = x(\alpha_1 - \beta_1 x) - k_1 xy$$

$$\frac{dy}{dt} = y(\alpha_2 - \beta_2 y) - k_2 xy$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, k_1, k_2$ are all positive constants. observe that our competition term is xy. Note, this is the simplest ~~form~~ type of competition for models of this form.

Let us find steady-state solution.

To find steady-state, set

$$\left.\begin{aligned} \frac{dx}{dt} = 0 \quad &\Rightarrow \quad x(\alpha_1 - \beta_1 x) - k_1 xy = 0 \\ \frac{dy}{dt} = 0 \quad &\Rightarrow \quad y(\alpha_2 - \beta_2 y) - k_2 xy = 0 \end{aligned}\right\} (*1)$$

Solve $(*1)$, to get the equilibria;

(i) $(0, 0)$

(ii) $\left(\frac{\alpha_1}{\beta_1}, 0\right)$

(iii) $\left(0, \frac{\alpha_2}{\beta_2}\right)$

(iv) $\left(\dfrac{k_1\alpha_2 - \beta_2\alpha_1}{k_1k_2 - \beta_1\beta_2} , \dfrac{\alpha_1 k_2 - \alpha_2\beta_1}{k_1k_2 - \beta_1\beta_2} \right)$

NB this steady-state only exist for $x>0$ and $y>0$.
(co-existence equilibrium).

Linearization of the system near the fixed points.

Let $\vec{Y}_0$ be a steady-state solution of the system. Then

$$\vec{F}(\vec{Y}_0) = \vec{0}$$

($\vec{u}(t)$ is very small)

Let $\vec{Y} = \vec{Y}_0 + \vec{u}(t)$, where $\vec{u}(t) << 1$.

put $\vec{Y}$ into $\vec{Y}' = \vec{F}(\vec{Y})$

$$\Rightarrow (\vec{Y}_0 + \vec{u}(t))' = \vec{F}(\vec{Y}_0 + \vec{u}(t))$$

$$\vec{u}'(t) = \vec{F}(\vec{Y}_0 + \vec{u}(t))$$

Since $\vec{u}(t) << 1$, we can Taylor expand $\vec{F}$ near $\vec{Y}_0$.

$$\Rightarrow \quad \vec{u}\,'(t) = \vec{F}(\vec{Y}_0) + D\vec{F}(\vec{Y}_0)\,\vec{u}(t) + \underbrace{\text{higher order term}}_{O(|\vec{u}|^2)}$$

(very small)

$$\Rightarrow \quad \vec{u}\,'(t) = \underbrace{\vec{F}(\vec{Y}_0)}_{=\vec{0}} + D\vec{F}(\vec{Y}_0)\,\vec{u}(t)$$

$$\Rightarrow \quad \vec{u}\,'(t) = D\vec{F}(\vec{Y}_0)\,\vec{u}(t)$$

$\therefore$ we have a homogeneous linear system.

where $D\vec{F}(\vec{Y}_0)$ is a matrix of partial derivatives and it is called the Jacobian matrix of $\vec{F}$ at $\vec{Y}_0$.

$$\cancel{D\vec{F}(\vec{Y}) = \Bigg(}$$

Let $\vec{F} = \begin{pmatrix} f(y_1, y_2) \\ g(y_1, y_2) \end{pmatrix}$

Then

$$D\vec{F}(\vec{Y}_0) = \begin{pmatrix} \left.\dfrac{\partial f}{\partial y_1}\right|_{\vec{Y}=\vec{Y}_0} & \left.\dfrac{\partial f}{\partial y_2}\right|_{\vec{Y}=\vec{Y}_0} \\ \left.\dfrac{\partial g}{\partial y_1}\right|_{\vec{Y}=\vec{Y}_0} & \left.\dfrac{\partial g}{\partial y_2}\right|_{\vec{Y}=\vec{Y}_0} \end{pmatrix}$$

The system $\vec{u}'(t) = D\vec{F}(\vec{Y}_0)\,\vec{u}$ describes the behaviour of the nonlinear system close to the steady-state solution $\vec{Y}_0$.

We shall use this system to study the behavior of the nonlinear system.

In particular, we shall use the eigenvalues and eigenvectors of the Jacobian matrix $D\vec{F}(\vec{Y}_0)$ to determine the stability of the equilibrium $\vec{Y}_0$.

Return to our competing model example:

$$\frac{dx}{dt} = x(3-4x) - xy$$

$$\frac{dy}{dt} = y(2-2y) - xy$$

Let $f(x,y) = x(3-4x) - xy$

$g(x,y) = y(2-2y) - xy$

Recall, the linearized system

$$\vec{u}\,'(t) = D\vec{F}(\vec{Y}_0)\,\vec{u}(t)$$

Let us construct $D\vec{F}(\vec{Y}_0)$

$$D\vec{F}(\vec{Y}) = \begin{pmatrix} 3 - \cancel{4}8x - y & -x \\ -y & 2 - 4y - x \end{pmatrix}$$

our steady-state solutions are

$\vec{Y}_1 = (0, 0)$, $\vec{Y}_2 = (3/4, 0)$

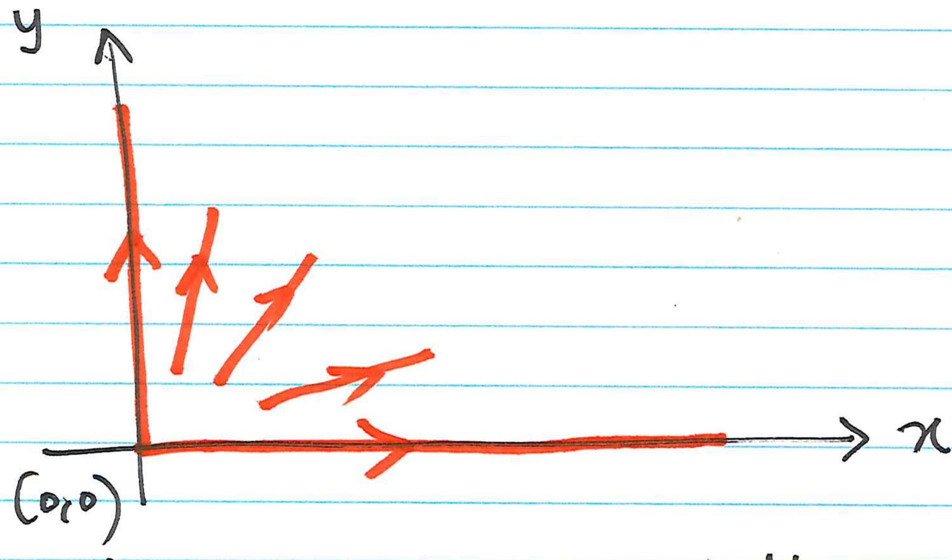
$\vec{Y}_3 = (0, 1)$, $\vec{Y}_4 = (4/7, 5/7)$

Near $\vec{Y}_1$,

$$D\vec{F}(\vec{Y}_1) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$\lambda_1 = 3$ and $\lambda_2 = 2$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



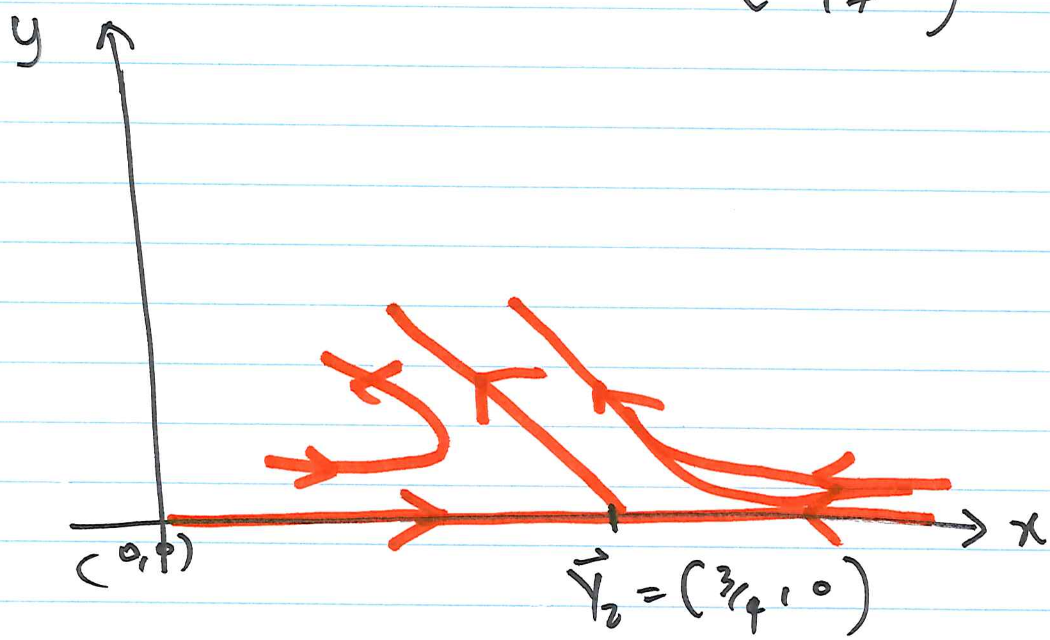
$\Rightarrow$ $\vec{Y}_1 = (0, 0)$ is <u>unstable</u>

Near $\vec{Y}_2 = (3/4, 0)$

$$D\vec{F}(\vec{Y}_2) = \begin{pmatrix} -3 & -3/4 \\ 0 & 5/4 \end{pmatrix}$$

$\lambda_1 = -3$ and $\lambda_2 = 5/4$

$\vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{V}_2 = \begin{pmatrix} -3 \\ 17 \end{pmatrix}$



$\Rightarrow \vec{Y}_2$ is a <u>saddle</u>

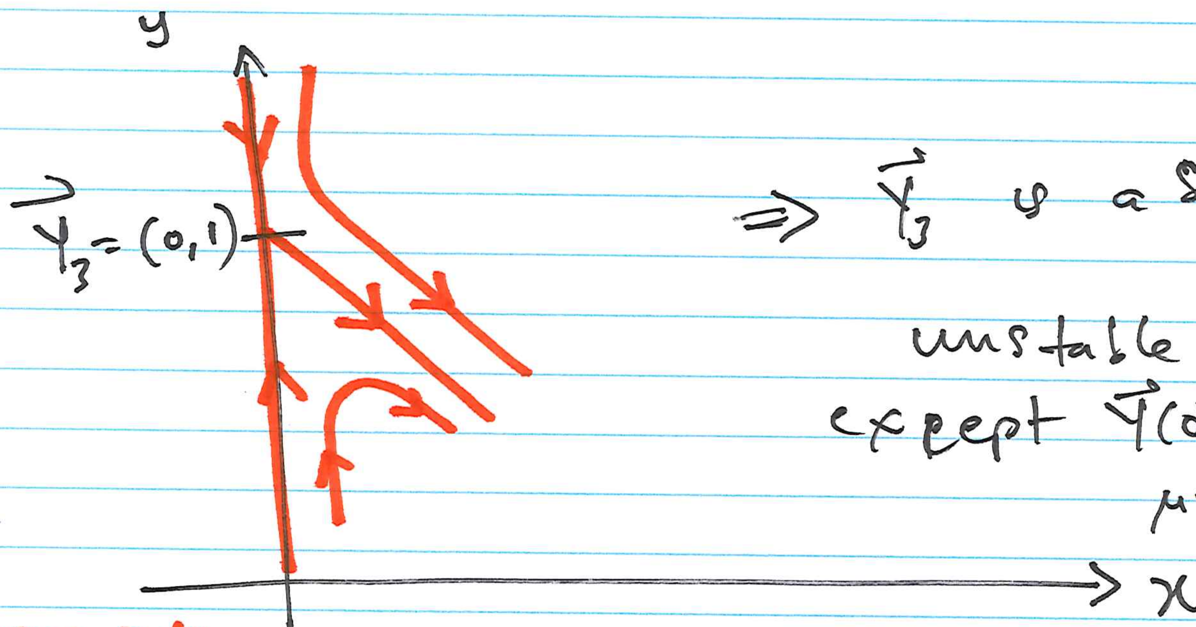
This equilibrium is unstable except we start a solution on the eigenvector $\vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Near $\vec{Y}_3 = (0, 1)$

$$D\vec{F}(\vec{Y}_3) = \begin{pmatrix} 2 & 0 \\ -1 & -2 \end{pmatrix}$$

$\lambda_1 = 2$ and $\lambda_2 = -2$

$\vec{V}_1 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $\quad\quad$ $\vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\Rightarrow \vec{Y}_3$ is a saddle

unstable

except $\vec{Y}(0) = (0, \mu)$

$\mu > 0$

** new **

Similarly, $\vec{Y}_3 = (0,1)$ is a unstable

except we start a solution on the eigenvector

$\vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For example: using the initial

condition $\vec{Y}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$. The solution decreases along

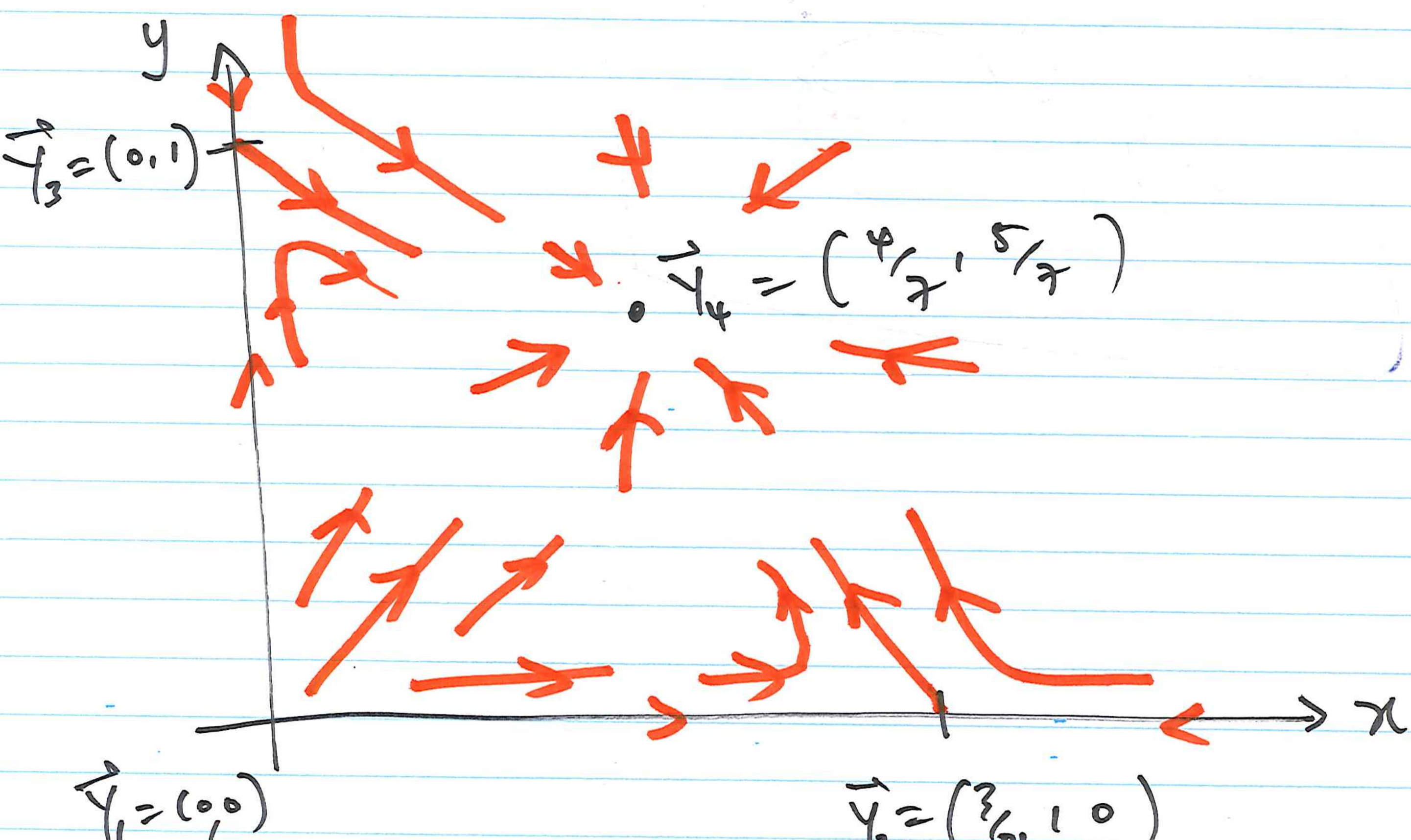
the eigenvector $\vec{V}_2$ and converges at $\vec{Y}_3$.

Near $\vec{Y}_4 = \left(\frac{4}{7}, \frac{5}{7}\right)$

$$D\vec{F}(\vec{Y}_4) = \begin{pmatrix} -16/7 & -4/7 \\ -5/7 & -10/7 \end{pmatrix}$$

$\lambda_1 = -2.6265$ and $\lambda_2 = -1.0878$

$\vec{v}_1 = \begin{pmatrix} 0.4306 \\ -0.9026 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} -0.8589 \\ -0.5122 \end{pmatrix}$



$\Rightarrow \vec{Y}_4$ is ~~attracto~~ stable