(1)

Math215/255 Section 104 Quiz 5 (15 Minutes)

Name: Solution Student Number:

November 7, 2017

Instructions: Attempt ALL questions.

Question One:

Consider the following second order ODE

$$y'' + y = t^3 + \sin(t) \tag{1}$$

(a) Write this equation as a system of first order equations. Show the your details.
(b) Find the homogeneous solution of the second order ODE.
(c) Suppose you want to find the particular solution of the ODE in Equation 1 using the method of undetermined coefficient, what will be your guess for y_p?

(a) Let $y_1 = y \Rightarrow y_1' = y' = y_2$

$y_2 = y' \Rightarrow y_2' = y''$

From (1), $y'' = -y_1 + t^3 + \sin(t)$

$\therefore$ $y_1' = y_2$

$y_2' = -y_1 + t^3 + \sin(t)$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ t^3 + \sin(t) \end{pmatrix}$$

(b) $y'' + y = 0$

$\Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$\therefore$ $y(t) = C_1 \cos(t) + C_2 \sin(t)$.

(c) ~~Observe~~ we have the ~~function~~ forcing function (2)

$g(t) = t^3 + \sin(t)$.

Observe that $\sin(t)$ is already in our homogeneous solution.

~~We write~~ $\therefore$ our guess for y_p will be

$$y_p(t) = At^3 + Bt^2 + Ct + D + Et\sin(t) + Ft\cos(t)$$

For t^3, we have

$$At^3 + Bt^2 + Ct + D$$

and for $\sin(t)$, we have

$$Et\sin(t) + Ft\cos(t)$$

$\therefore$

$$y_p = At^3 + Bt^2 + Ct + D + Et\sin(t) + Ft\cos(t).$$

Question Two:

3

Consider the forced system

$$\vec{Y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{Y} + \vec{g}(t),$$

where $\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and $\vec{g}(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$.

(a) Find the homogeneous solution of the system.

(b) Use the method of undetermine coefficient to find the particular solution of the system.

(c) Use $\vec{Y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to find the constant in the general solution of the system.

(a) For the homogeneous solution, we solve

$$\vec{y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{Y}$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$\Rightarrow \lambda_1 = -3$ and $\lambda_2 = -1$ are the eigenvalues.

For $\lambda_1 = -3$, $(A - \lambda_1 I)\vec{V}_1 = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -1$, $(A - \lambda_2 I)\vec{V}_2 = 0$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \vec{Y}_H = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

(b) We have $\vec{g}(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t}$

Observe that e^{-t} is already in our homogeneous solution.

Now, if we relate this to the case of having repeated eigenvalues for system of equations, we would see that our guess for the particular solution for the term in $\vec{g}(t)$ that has e^{-t} will be

$$\vec{Y}_{p_1} = (\vec{A}t + \vec{B})e^{-t}$$

and for the term with t, we have

$$\vec{Y}_{p_2} = \vec{C}t + \vec{D}$$

put these together, we have our guess to be

$$\vec{Y}_p = \vec{Y}_{p_1} + \vec{Y}_{p_2}$$

$$\vec{Y}_p = (\vec{A}t + B)e^{-t} + \vec{C}t + \vec{D}$$

$$\vec{Y}_p' = \vec{A}e^{-t} - \vec{A}te^{-t} - Be^{-t} + \vec{C}$$

put $\vec{Y}_p$ into

$$\vec{Y}_p' = A\vec{Y}_p + \vec{g}(t), \text{ where } A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

We have

$$-\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} te^{-t} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-t} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2a_1 + a_2 \\ a_1 - 2a_2 \end{pmatrix} te^{-t} + \begin{pmatrix} -2b_1 + b_2 \\ b_1 - 2b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} -2c_1 + c_2 \\ c_1 - 2c_2 \end{pmatrix} t$$

$$+ \begin{pmatrix} -2d_1 + d_2 \\ d_1 - 2d_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t$$

collecting coefficients,

For te^{-t}, we have

$$\begin{pmatrix} -2a_1 + a_2 \\ a_1 - 2a_2 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix} \Rightarrow \vec{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For e^{-t}, $\begin{pmatrix} -2b_1 + b_2 \\ b_1 - 2b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -b_1 + b_2 \\ b_1 - b_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \vec{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

For t, $\begin{pmatrix} -2c_1 + c_2 \\ c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \Rightarrow \vec{C} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For constants,

$$\begin{pmatrix} -2d_1 + d_2 \\ d_1 - 2d_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \vec{D} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix}$$

(b)

$$\therefore \vec{y}_p = \begin{pmatrix}1\\1\end{pmatrix} t e^{-t} + \begin{pmatrix}0\\-1\end{pmatrix} e^{-t} + \begin{pmatrix}1\\2\end{pmatrix} t - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}$$

$$\vec{y}_p = \left(\begin{pmatrix}0\\-1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} t \right) e^{-t} + \begin{pmatrix}1\\2\end{pmatrix} t - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}$$

Thus, the general solution

$$\vec{y} = c_1 \begin{pmatrix}1\\-1\end{pmatrix} e^{-3t} + c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{-t} + \left(\begin{pmatrix}0\\-1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} t \right) e^{-t}$$
$$+ \begin{pmatrix}1\\2\end{pmatrix} t - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}$$

(c) $\vec{y}(0) = \begin{pmatrix}0\\0\end{pmatrix}$,

$$c_1 \begin{pmatrix}1\\-1\end{pmatrix} + c_2 \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}4/3\\8/3\end{pmatrix}$$

$\cancel{c_2 = 2 \cdot 4/3}$ $\quad c_1 = -2/3$ and $c_2 = 2$.